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LAW OF FACILITY OF ERRORS IN TWO DIMENSIONS.

BY E. L. DE FOREST.

[Continued from page 48.]

THIS surface (43) might be called the *probability surface*, just as (1) is the probability curve. Its form is such that if it is intersected by a plane parallel to the plane of XY, the lines of intersection will be ellipses whose centre and axes, projected on the plane of XY, coincide with the origin and axes of X and Y, and the axes of the ellipse in the x and y directions are inversely proportional to h_1 and h_2, and therefore directly proportional to r_1 and r_2. This may be shown by making z a constant in (43). The projected ellipses are the loci of points of equal probability.

From this symmetry of the surface with respect to the axes of X and Y, it follows that these are the free axes of the system of expanded coefficients z, regarded as masses in the plane of XY. Thus the given probabilities λ, and the resultant probabilities z, have both the same free axes. This also follows from the permanence of direction of the axes which give maximum and minimum values to the radii of gyration in successive powers, as already noticed. The value of z is a maximum at the origin, and this being the place of the centre of gravity of the whole system of masses z, it appears that the point of most probable error in the result of a very large number of observations, will be located at the centre of gravity of all the points which can possibly result under the given system of elementary errors, each possible point of error being taken with a weight proportional to the probability that such error will occur. This also follows from what we have already proved in connection with formula (3). In order that the most probable result should coincide with the true point or place of zero error, it would be necessary and sufficient that the probabilities or weights λ of the elementary points of error should be such that their centre of gravity falls at the zero point.

Suppose the plane of XY to be divided up into elementary rectangles $dxdy$ by lines drawn parallel to the axes; then (43) gives the probability z that an error which occurs will fall on the rectangle whose centre is at the point $x = idx$, $y = jdy$. It gives an approximate result when instead of dx and dy we use small finite distances Δx and Δy. The values of h_1 and h_2 are computed by (37) and (38) if the elementary errors and their probabilities λ are known. But if these are unknown, as generally happens in practical applications, we must compute from a large number of observed errors, the quadratic mean errors in the x and y directions, answering to $r_1\sqrt{k}$ and $r_2\sqrt{k}$ in (38), and thus obtain h_1 and h_2. For example, if a large number of shots have been fired and have struck a target, and we wish to infer, from the distribution of the shot-marks, what is the law of probable distribution of other shots to be made under like conditions, our investigation leads to the following process. Regarding each given shot-mark as the mass of a material point, all such masses being equal, find the centre of gravity of the system, and then by (31) find its free axes. The distance of the centre of gravity from any assumed axis of reference is the arithmetical mean of the distances of all the shot-marks from that axis, distances on opposite sides of it having contrary signs. The free axes will be at right angles to each other at the centre of gravity, and we take either one of them as the X and the other as the Y axis. Find the quadratic mean errors in the x and y directions, that is, the square root of the mean of the squares of the distances of the shot-marks from the Y and X axis respectively. Denoting these by e_1dx and e_2dy, we shall have as in (38),

$$(h_1dx)^2 = 1 \div 2e_1^2, \qquad (h_2dy)^2 = 1 \div 2e_2^2, \tag{44}$$

and (43) becomes

$$z = \frac{1}{2\pi e_1 e_2} e^{-(i^2 \div 2e_1^2)-(j^2 \div 2e_2^2)}. \tag{45}$$

For convenience, dx and dy may be taken equal to each other and to any small unit of measure, for instance one inch. Then (45) gives the probability z that any particular future shot will strike that square inch of the target whose coordinates are i and j inches.

All this is on the supposition that we know nothing about the probable distribution of the errors except what is to be inferred from the actual distribution of the shot-marks already made.

The principle that the axes of X and Y should be taken to coincide with the free axes of the given shot-marks, is one which has not, so far as I know, been brought out by any previous writer. The usual course has been, to draw the axes horizontally and vertically through the centre of gravity of the marks. This, however, will in general be correct only when there is reason to believe that the elementary errors, whose probabilities are λ, are distributed so as to make the horizontal and vertical axes free axes, which will be the case for instance when these errors and their probabilities fall symmetrically on either side of the horizontal and vertical axes, making them axes of symmetry in the system of masses λ. Granting that the free axes are horizontal and vertical, formula (43) agrees with the one in general use. It can be written

$$z = \left(\frac{h_1 dx}{\sqrt{\pi}} e^{-h_1^2 x^2}\right)\left(\frac{h_2 dy}{\sqrt{\pi}} e^{-h_2^2 y^2}\right), \tag{46}$$

the second member being the product of two factors of the same form as in the probability curve (1). This property has enabled writers to give a demonstration of (43), or what has seemed to be one, in a very simple manner. (See for instance Didion's *Calcul des Probabilités appliqué au Tir des Projectiles*, Paris, 1858, p. 42.) If we suppose the plane of XY to be divided by lines drawn parallel to the Y axis, into bands of the width dx, and assume that the probability of deviation from the Y axis is the same at all parts of that axis, or in other words, is independent of y, then the probability that an error which occurs will fall within the band whose distance from the Y axis is x, is represented by the first of the two factors in (46). Likewise if lines are drawn parallel to the X axis, forming bands of the width dy, the probability that the error will fall within the band whose distance from the X axis is y, is represented by the second factor in (46). But the probability that two independent events will both occur, is the product of their separate probabilities, so that (46) gives the probability z that the error will fall within both bands, that is, at their intersection, and within the rectangle $dx\,dy$ whose coordinates are x and y. This demonstration, however, has been objected to, on the ground that the independence of the x and y deviations, if it exists, is not so evident as to be axiomatic. It really seems to be a thing to be proved. Various authors who have discus'd the point are named in Merriman's *List of Writings relating to the Method of Least Squares.* (Trans. Conn. Acad., Vol. IV, Part 1.) Our present investigation shows that the x and y deviations may be regarded as independent of each other, provided that the axes of X and Y are taken to coincide with the free axes of the probabilities λ of the elementary errors. When these errors are unknown, we approximate to the desired axes by taking the free axes of the observed points of error, or shot-marks. But if the axes in (43) are supposed to be rotated through the angle θ, and the coordinates to these new axes are x' and y', so that we have

$$x = x' \cos\theta - y' \sin\theta, \qquad y = x' \sin\theta + y' \cos\theta,$$

then substituting these for x and y in (43), the exponent of e will contain not only x'^2 and y'^2, but the product $x'y'$, so that it will not in general be possible to separate the second member into two factors, one containing the abscissa and the other the ordinate, as in (46). Thus the x and y deviations are not in general independent of each other unless the axes taken are free axes. But if we assume that the elementary errors are such as to make given deviations equally probable in all directions, every axis through the centre of gravity will be a free axis, and the deviations in horizontal and vertical directions will therefore be independent. That assumption is ordinarily made in practice, as it simplifies the calculations. The deviations of the shot-marks are then measured from the centre of gravity directly, instead of from the axes. Denoting the square root of the mean of the squares of these deviations by εdx, and taking $dx = dy$, we have

$$\varepsilon^2 = e_1^2 + e_2^2,$$

and since e_1 and e_2 are here equal, $\varepsilon^2 = 2e_1^2 = 2e_2^2$, and (44) gives

$$(h_1 dx)^2 = (h_2 dy)^2 = 1 \div \varepsilon^2, \quad \therefore h_1 = h_2 = 1 \div (\varepsilon dx).$$

(Compare Didion's work already cited, and Sonnet's *Dictionnaire des Mathématiques Appliquées*, article *Probabilité du Tir.*) The probability surface is what (43) becomes when $dx = dy$, $h_1 = h_2$ and $x^2 + y^2 = r^2$, the elliptical sections being reduced to circles. It has been found to agree sufficiently well with the results of observation in target shooting. Probably it will not be worth while to compute the place of the free axes by (31) unless the observed points of error or shot-marks form a decidedly elongated group. If the probability of deviation is really the same in all directions, still it is not likely that the actual distribution of the shots will be uniform, and a slight elongation of the group will not compel the inference that the elementary errors must be greater in that direction.

Returning now to the strict construction of the surface (43), let us refer the position of the rectangle $dx\,dy$ to polar coordinates by writing

$$x = r \cos \varphi, \qquad y = r \sin \varphi, \tag{47}$$

so that we have

$$z = \frac{h_1 h_2 dx\,dy}{\pi} e^{-r^2 (h_1^2 \cos^2 \phi + h_2^2 \sin^2 \phi)} \tag{48}$$

as the probability that an error which occurs will fall on the area $dxdy$ whose coordinates are r and φ. Suppose that the plane of XY is divided into rings by circles drawn from the origin as a centre, the breadth of each ring being $\sqrt{(dxdy)}$, and that it is also divided into sectors by radii drawn at angular intervals equal to $d\varphi$. The plane is thus partitioned into small elementary areas, each area at distance r from the origin being equal to $rd\varphi \times \sqrt{(dx\,dy)}$, and $dx\,dy$ is contained in it $rd\varphi \div \sqrt{(dx\,dy)}$ times, so that the probability that an error which occurs will fall on this area, is

$$zrd\varphi \div \surd(dxdy).$$

Hence the probability that it will fall on a given sector whose extreme radius is r will be

$$p = \frac{1}{dr}\int_0^r z\left(\frac{rd\varphi}{\surd(dxdy)}\right)dr,$$

where dr is to be taken equal to $\surd(dxdy)$; so that substituting the value of z from (48), we have

$$p = \frac{h_1h_2d\varphi}{\pi}\int_0^r e^{-r^2(h_1^2\cos^2\phi+h_2^2\sin^2\phi)}\,rdr,$$

and by integration

$$p = \frac{h_1h_2d\varphi}{2\pi(h_1^2\cos^2\varphi+h_2^2\sin^2\varphi)}\left\{1 - e^{-r^2(h_1^2\cos^2\phi+h_2^2\sin^2\phi)}\right\}. \qquad (49)$$

Making $r = \infty$ we get as the probability that the error will fall somewhere on this sector extended to infinity,

$$\rho = \frac{h_1h_2d\varphi}{2\pi(h_1^2\cos^2\varphi + h_2^2\sin^2\varphi)}. \qquad (50)$$

If now it is required that $p = \frac{1}{2}\rho$, we have the relation

$$e^{-r^2(h_1^2\cos^2\phi+h_2^2\sin^2\phi)} = \tfrac{1}{2},$$

and restoring x and y from (47),

$$e^{-(h_1^2x^2+h_2^2y^2)} = \tfrac{1}{2}, \qquad \therefore\ h_1^2x^2+h_2^2y^2 = (\log 2)\div(\log e).$$

Giving to h_1 and h_2 their values as in (44),

$$h_1^2 = 1\div 2(e_1dx)^2, \qquad h_2^2 = 1\div 2(e_2dy)^2,$$

we get the equation of an ellipse,

$$\frac{x^2}{[(\log 4)\div(\log e)](e_1dx)^2} + \frac{y^2}{[(\log 4)\div(\log e)](e_2dy)^2} = 1,$$

whose semi-axes are the square roots of the denominators in the first member. Their values are

$$a = 1.17741(e_1dx), \qquad b = 1.17741(e_2dy), \qquad (51)$$

so that they are proportional to e_1dx and e_2dy, the quadratic mean errors in the x and y directions. Hence for example, in the group of shot-marks already considered, if we find the free axes and the quadratic mean deviations from them, and construct an ellipse with semi-axes a and b as in (51) coinciding with the free axes of X and Y respectively, this will be the *ellipse of probable error*, in the sense that it is an even chance whether a particular future shot will strike inside or outside the ellipse. And if two straight lines are drawn from its centre making any angle with each other, it is an even chance whether shots which fall between them will fall inside or outside of the ellipse.

There is a correspondence between some of our results and those which have been reached by Andrä and others in the theory of the error-ellipse (*die Fehlerellipse*), which is one of the more recent developm'ts of the method of least squares as applied to the location of a point by triangulations. The true position of the point should be at the common intersection of several straight lines, whose directions are given free from error, but whose positions laterally are subject to errors of observation, so that the lines do not in general have a common intersection. Each of them may be represented by an equation of the form

$$ax + by + l = 0,$$

where x and y are the coordinates of any point in the line, referred to rectangular axes taken in the vicinity of the desired point, a and $-b$ are the sine and cosine of the acute angle which the line makes with the X axis, on the $+X$ side, and l is the perpendicular from the origin upon the line. The angle and the perpendicular are essentially $+$ or $-$ according as thay lie above or below the X axis. The various intersecting lines furnish a number of observation equations of this form, which being solved by the method of least squares, give the desired coordinates x and y of the most probable point of common intersection. It is then shown that all other points whose probabilities are equal are ranged about the most probable point in ellipses which are similar and similarly situated and concentric. The directions of their major and minor axes are given by a formula which bears an external resemblance to (31). See for instance Jordan's *Handbuch der Vermessungskunde*, I., pp. 120 and 123.

The dimensions of the ellipse within which it is an even chance that the true point will fall, have been ascertained by Helmert. (*Ausgleichungsrechnung*, p. 236.) He makes its semi-axes bear to the corresponding quadratic mean errors of the point, the same ratio 1.17741 which I have obtained in a different way in (51). Hence they bear to the corresponding probable errors the ratio

$$1.17741 \div .6745 = 1.7456.$$

His demonstration will be found in Schlömilch's *Zeitschrift fur Math. und Phys.*, 1868, pp. 75 to 79. The probability of a given deviation of the p't from its most probable place, is expressed by the same form of function which I have found as the equation (43) of the probability surface.

It may be questioned, however, whether the theory of the error-ellipse affords proof of the universality of that law of facility, for all accidental errors in the situation of a point in a plane. It seems to me that when taken in this extended sense, the method is open to the same kind of theoretical objection which we noticed in connection with (46). An observed point of error, or intersection of two erroneous observed lines, is the result of the concurrent happening of two independent errors, each of which, it is assumed by the theory, follows an exponential law of facility such as is expressed by the probability curve (1). The probability that any particular point in the plane is the true one, is regarded as proportional to the probabilty of the concurrent happening of all the independent errors represented by the perpendicular distances from that point to all of the observed lines. The most probable point is that which renders the sum of the squares of the distances, each square being multiplied by the weight due to its observed line, a minimum. (Helmert, in the *Zeitschrift*, pp. 76 and 89; also Jordan, II:, p. 223.)

For the case of intersecting straight lines which lie in only two mutually perpendicular directions, the demonstration is evidently similar in principle to that which I remarked upon in connection with (46). My own proof of (43), however, is a direct proof, which does not take (1) into account at all, inasmuch as it does not assume that the x- and y- errors of a point happen independently of each other. But it includes the proof of (1) as a special case, for if the elementary points of error are all on one line which is made the axis of X, so that the given polynomial (2) is reduced to one dimension, all its coefficients being zero except those on the horizontal line through the middle of the square group, the last equation in (23) disappears, and of the two equations (30) only the first one remains, with $\alpha_2 = 0$ and $\gamma = 0$. When the centre of gravity of the masses λ is made the origin and centre of the linear group, by imagining terms with zero coefficients to be added if necessary, we get $\alpha_1 = 0$, and only the first of the two differential equations will exist in (35) and (39), the integral being

$$z = ce^{-h_1{}^2x^2}.$$

The condition (41) is reduced to

$$\frac{1}{dx}\int_{-\infty}^{+\infty} z dx = 1, \qquad \therefore c = \frac{h_1 dx}{\sqrt{\pi}},$$

and z becomes the ordinate to the probability curve, like y in (1).

The occurrence of errors in two dimensions can be further illustrated by the drawing of balls from an urn, each ball being marked with two numbers which may be positive or negative, the first number representing the x and the second the y coordinate of the point of error, reckoned from the true point, or place of error zero, as an origin. Each ball is replaced as soon as drawn, and they are all mixed so as to keep the chances the same.

Suppose for example that the urn contains 20 balls marked as follows; one of them is marked 0, 0, four are marked 1, 0, five 1, 1, four 0, 1, one —1, 1, three —1, —1, and two 0, —1. Then if k drawings are made, the probability that the algebraic sum of the first or x-numbers on the balls drawn will be t, and that at the same time the sum of their sec'd or y-numbers will be v, is the coefficient of $\xi^t\eta^v$ in the expansion of the polynomial (52) to the k power, this polynomial being a case under the general form (2).

$$\frac{1}{20}\left(\begin{array}{rrr} \xi^{-1}\eta & +4\eta & +5\xi\eta \\ & +1 & +4\xi \\ +3\xi^{-1}\eta^{-1} & +2\eta^{-1} & \end{array}\right) \tag{52}$$

To find the centre of gravity and the free axes of the coefficients λ, we may disregard for the present the divisor 20 common to them all, and write them in the rectangular form (53). Their centre of gravity is at the point whose coordinates reckoned from the middle term 1, are $u' = \frac{1}{4}\varDelta u$, $v' = \frac{1}{4}\varDelta v$, where $\varDelta u$ and $\varDelta v$ are the intervals between the terms in (53), horizontally and vertically. Drawing horizontal and vertical axes of U and V through this centre of gravity, the abscissas u of the terms in the three columns are

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 0 & 1 & 4 \\ \hline 3 & 2 & 0 \\ \hline \end{array} \tag{53}$$

$$-\tfrac{5}{4}\varDelta u, \qquad -\tfrac{1}{4}\varDelta u, \qquad \tfrac{3}{4}\varDelta u,$$

and the ordinates v of the terms in the three rows are

$$\tfrac{3}{4}\varDelta v, \qquad -\tfrac{1}{4}\varDelta v, \qquad -\tfrac{5}{4}\varDelta v.$$

Denoting the terms in (53) by M, and applying (31), we find

$$\Sigma(Muv) = \tfrac{23}{4}\varDelta u\varDelta v, \qquad \Sigma(Mu^2) = \tfrac{47}{4}(\varDelta u)^2, \qquad \Sigma(Mv^2) = \tfrac{55}{4}(\varDelta v)^2.$$

$$\tan 2\varphi = 46\left(\frac{\varDelta u}{\varDelta v}\right) \div \left\{ 47\left(\frac{\varDelta u}{\varDelta v}\right)^2 - 55 \right\}.$$

Taking for convenience $\varDelta u = \varDelta v$, so that (53) is a square, we have $\tan 2\varphi = -\frac{23}{4}$, wherefore $\varphi = -40°4'$. This is the angle which one free axis makes with the horizontal, at the centre of gravity, and the second free axis is at right angles to the first. Designating these axes as X and Y, we get from (33) the relation between the new coordinates and the old,

$$x = u\cos\theta - v\sin\theta, \qquad y = u\sin\theta + v\cos\theta, \tag{54}$$

where $\theta = -\varphi = 40°4'$. Squaring x and y, multiplying them by M, and summing the results for all the terms M in the system, we have

$$\left.\begin{array}{l} \Sigma(Mx^2) = \Sigma(Mu^2)\cos^2\theta + \Sigma(Mv^2)\sin^2\theta - 2\Sigma(Muv)\sin\theta\cos\theta, \\ \Sigma(My^2) = \Sigma(Mu^2)\sin^2\theta + \Sigma(Mv^2)\cos^2\theta + 2\Sigma(Muv)\sin\theta\cos\theta. \end{array}\right\} \tag{55}$$

We assign to the sums in the second members their values already found, and take for simplicity

$$\varDelta x = \varDelta y = \varDelta u = \varDelta v, \tag{56}$$

and so get

$$\Sigma(Mx^2) = 6.909(\varDelta x)^2, \qquad \Sigma(My^2) = 18.588(\varDelta y)^2.$$

Dividing these by 20, since $\lambda = M \div 20$, we obtain the squares of the radii of gyration of the masses λ about the Y and X axes,

$$r_1^2 = .3454(\varDelta x)^2, \qquad r_2^2 = .9294(\varDelta y)^2.$$

Suppose that the number of drawings from the urn is $k=4$. Then in the expanded polynomial the squares of the radii of gyration of the coeff's are

$$kr_1^2 = 1.382(\Delta x)^2, \qquad kr_2^2 = 3.718(\Delta y)^2,$$

so that in (44) we have

$$e_1^2 = 1.382, \qquad e_2^2 = 3.718,$$

and (45) becomes

$$\left.\begin{aligned} z &= \phantom{\bar{2}}.0702e^{-.362i^2-.134j^2}, \\ \therefore \log z &= \bar{2}.8464 - .15712i^2 - .05840j^2. \end{aligned}\right\} \qquad (58)$$

This is the equation of the limiting surface which represents approximately the contour of the coefficients in the expansion of (52) to the fourth power. Since the centre of gravity preserves in the expansion the same relative position which it had in the first power, its coordinates reckoned from the middle point will be four times as great, or

$$u' = \Delta u, \qquad v' = \Delta v.$$

The expansion contains 81 terms, wich occupy the centres of the small squares in the subjoined figure, and the centre of gravity falls at the next term diagonally from the middle one. The axes of U and V have their origin at this point, as do also those of X and Y. From the coordinates u and v for the several terms we compute the values of x and y by (54). A few nearest the origin being tabulated as in (59), the rest can be readily found from them by their differences. The values of z are now computed, and being multiplied by 1000, they are as shown in the first half of the accompanying table, omitting decimals. Also by actual expansion of (52) to the 4th power we get the true coefficients, which being multiplied by 1000 like the others, are as shown in the second half of the table. The general agreement between the two sets of coefficients is evident, though it is not very close because $k = 4$ is not a large number. It will be seen that the largest tab-

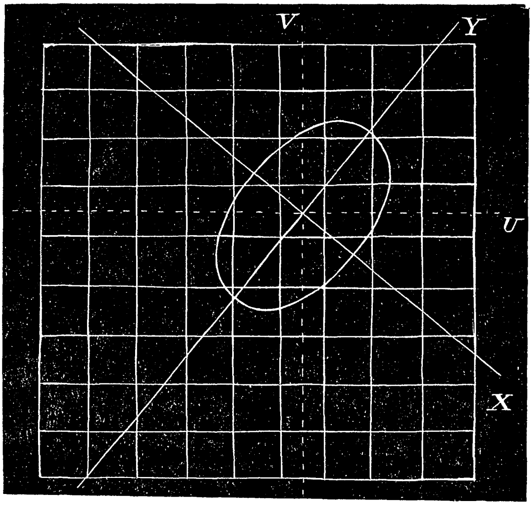
$i = x \div \Delta x$			$j = y \div \Delta y$		
—1.409	—.644	.122	.122	.765	1.409
— .765	0	.765	— .644	0	.644
— .122	.644	1.409	—1.409	—.765	— .122

(59)

(1000z).

0	0	0	1	4	9	13	12	6	0	0	1	3	9	15	18	13	4
0	0	1	4	14	28	34	24	10	0	0	0	3	11	26	39	33	12
0	0	3	12	34	56	53	30	10	0	1	5	18	37	52	53	38	15
0	1	6	24	54	70	54	24	6	0	0	3	15	42	66	58	27	8
0	2	10	30	53	56	34	12	3	0	3	13	29	44	49	38	14	2
0	2	10	24	34	28	14	4	1	0	1	6	22	36	29	13	3	0
0	2	6	12	13	9	4	1	0	1	4	10	14	14	9	2	0	0
0	1	2	4	3	2	1	0	0	0	1	4	6	4	1	0	0	0
0	0	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0

ulated numbers form groups elongated in the direction of the free axis of Y. In accordance with (56), the numbers should properly be arranged in a square as in the figure.



The position of the axes of UV and of XY are there shown, and also the ellipse of probable error. The semi-axes of this ellipse as given by (51) and (57), are

$$a = 1.38dx,$$
$$b = 2.27dy,$$

where $dx = dy$ is the side of one of the small squares in the figure.

The ellipse encloses rather more than nine squares, allowing for fractions, but we can see that it is not far from an even chance that the algebraic sum of the x- and y-numbers on the balls drawn from the urn will be some one of the nine pairs.

$$\begin{array}{ccc} 0,2 & 1,2 & 2,2 \\ 0,1 & 1,1 & 2,1 \\ 0,0 & 1,0 & 2,0 \end{array}$$

The most probable result is 1, 1. By following the analogy of this example we can construct an equation such as (58) to represent the expansion of any given polynomial, and to show the distribution of the errors resulting, in a given number of trials, from any set of elementary errors we please to assume.

It should be noticed, however, that the coefficients of a polynomial representing probabilities are all positive, whereas the general problem, to represent the limiting form of an expansion when some of the coefficients are negative, may give rise to special cases not included in our present investigation. The limiting surface, if there is one, will in gen'l be the probab'ty surface (43), but not always. In (29) it may happen that β_1 or β_2, or both of them, are zero, so that the differential equations (35) vanish, and our process fails. In such a case it is no longer sufficient to consider, as we did in connection with (26), that the element of the surface is a plane surface. Its curvature must be taken into account, and the limiting surface will most likely consist of an infinite series of undulations, analogous to those of the higher curves discussed in my former article. (See ANALYST, Sept. '79)